Coupled Oscillations in Diverse Phenomena

Part 3: Neutrino oscillations

by Dwight E. Neuenschwander, Physics Professor, Southern Nazarene University

Parts 1 and 2 of this article appeared in the Spring and Fall 2022 issues of Radiations *and are available at www.sigmapisigma.org/sigmapisigma/radiations/archive. Numbering in Part 3 continues from the previous parts.*

Editor's note: We inadvertently created a new—and meaningless—matrix notation when we published Part 2. Our sincere apologies to author Dwight Neuenschwander and to you, our readers. The version available on our website has been corrected.

The Backstory

The neutrino ν ("little neutral one") was postulated by Wolfgang Pauli in 1930 to save the principles of energy and angular momentum conservation in beta decay. Without knowing about the neutrino, an observed reaction seemed to be $n \rightarrow p + e^-$. Consider energy: for the decay of a free neutron, in the neutron's rest frame the conservation of energy gives

$$m_n c^2 = \left(m_p + m_e\right)c^2 + K_e + K_p \tag{44}$$

where *K* denotes kinetic energy, and by conservation of momentum [6],

$$p_p = p_e. \tag{45}$$

Equations (44) and (45) together imply a unique kinetic energy for the emitted electron. Most neutrons reside in nuclei, and for the beta decay that turns, say, boron-12 into carbon-12, in the absence of neutrinos the unique kinetic energy of the electron would be 13.37 MeV. However, the actual kinetic energies of the electrons emitted in boron-12 beta decay range across the continuum from zero to 13.37 MeV. As Pauli realized, the presence of a third emitted particle offers an infinite number of ways to partition the kinetic energy and momentum, allowing the continuum of electron kinetic energies while preserving energy conservation. This third particle must have zero charge, since electric charge is conserved and the charges are already accounted for. Furthermore, it would have to interact ever so weakly since it slipped under the detection threshold of original instrumentation, suggesting the neutrino has very little if any mass. (According to neutrino reaction cross sections eventually measured, if a beam of neutrinos was sent through a block of lead a light-year thick, most of them would emerge out the other side!) The very weakness of neutrino interactions allows observations so far to give only upper bounds to the particle's observables, such as its mass and magnetic dipole moment; typically, these quantities are taken to be zero, which, if not exactly correct, are good approximations.

The neutrino (actually the antineutrino $\bar{\nu}$) was first detected in 1956 when a team led by Frederick Reines and Clyde Cowan took advantage of the high antineutrino flux of the new Savannah River nuclear power plant to compensate for the particle's low-probability interactions. Beta decays coming from the byproducts of the fission include the abundant reaction

$$p \rightarrow n + e^- + \bar{\nu}.$$

Then and now, neutrinos and antineutrinos are detected indirectly by detecting the particles they produce in reactions. According to theory, the antineutrino from beta decay can drive the reaction,

$$\bar{\nu}_e + p \rightarrow n + e^+.$$

The position immediately annihilates with an electron to produce two 0.511 MeV gamma-ray photons, and in the Reines-Cowan experiment the neutron is absorbed by cadmium (with which the newly developed organic liquid scintillator detector was spiked) to produce a 9 MeV photon. The delay of a few microseconds between the production of the 0.511 MeV and 9 MeV photons allows a delayed-coincidence detection of the photons through an array of photomultiplier tubes [7].

As we presently understand the zoo of elementary particles, the fermions that participate in the strong and weak forces are the quarks, and the fermions that do not participate in the strong force are the leptons, whose most common representatives are the electron and its corresponding neutrino. There are three "flavors" (as they are whimsically called) of quarks and leptons; only the leptons concern us here [8], and the three lepton flavors are the electron e^- and its neutrino ν_e; the muon μ^- and its neutrino ν_μ; and the tau τ^- and its neutrino ν_τ—and their antiparticles. They all carry spin ½; the electron, muon, and tau carry negative charge (their antiparticles carry positive charge). The electron's mass is about ½ MeV/c^2, the muon mass about 106 MeV/c^2, and the tau mass about 1777 MeV/c^2. Lepton flavor is so far evidently conserved (approximately if not exactly so) [9]; electron-type neutrinos go with electrons, muon neutrinos with muons, and tau neutrinos with taus. The appearance of neutrinos is the hallmark of the weak interaction in reactions such as

$$n \to p + e^- + \bar{\nu}_e \tag{46a}$$

$$\nu_\tau + n \to p + \tau^- \tag{46b}$$

$$\mu^- + p \to \nu_\mu + n \tag{46c}$$

and so on.

The Solar Neutrino Problem

In the mid-1960s the famous "solar neutrino problem" surfaced. In the sun's core electron-type neutrinos are produced in the first step of the proton-proton cycle of nuclear fusion, $p + p \to H_1^2 + e^+ + \nu_e$. Since neutrinos interact so weakly, they escape the sun at once, and each one carries no more than about 20 MeV of energy. Some of them pass through Earth. The neutrino detector in the Homestake Mine in South Dakota first raised the alarm about the solar neutrino problem. Headed by Raymond Davis, the experiment, which ran from 1970 to 1994, featured a large tank of dry-cleaning fluid, carbon tetrachloride CCl_4. It detects solar neutrinos through the reaction

$$\nu_e + n \to p + e^-. \tag{46d}$$

When this reaction occurs in a chlorine nucleus, it becomes the nucleus of the noble gas argon, which bubbles out to be collected. After several years of data collecting, the observed flux of neutrinos was about 2/3 short of the prediction, even when the solar models took into account other fusion channels such as the CNO cycle in addition to the *p-p* cycle.

The fact that neutrinos from the sun carry less than 20 MeV but the muon and tau masses are 106 and 1777 MeV/c^2, respectively, means that even if the neutrino could, somehow, produce the reaction, $\nu + n \to p + \mu^-$; in other words, if muon neutrinos were somehow in the mix of incoming solar neutrinos, they would still not have enough energy to make the muon appear in the chlorine-to-argon reaction. The insensitivity of the CCl_4 to muon neutrinos was suggestive. The most promising solution, first published by Bruno Pontecorvo in 1957 [10], could explain the discrepancy while allowing conservation of lepton number and lepton flavor in interactions (particles and antiparticles have opposite signs of these numbers). Pontecorvo suggested the possibility that a neutrino oscillates between flavors as it travels freely, not interacting with anything! In other words, a ν_e traveling freely might change into a ν_μ, then into a ν_τ, or back into a ν_e. *This is a coupled oscillator problem*. The coupling and the frequency of this oscillation depend on two neutrino species having a mass *difference.*

To model this business mathematically, the moment of epiphany comes with realizing that when *propagating freely* the neutrino state is a *mass* eigenstate, but when *interacting* the neutrino state is a *flavor* eigenstate. The flavor eigenstates are what we mean when saying a neutrino is an electron neutrino ν_e, or a muon neutrino ν_μ, or a tau neutrino ν_τ. The mass eigenstates are superpositions of the flavor eigenstates, and, conversely, the flavor eigenstates are superpositions of mass eigenstates—which means that "the" mass of a ν_e or ν_μ or ν_τ particle is not well defined. Be that as it may, the mass and flavor eigenstates form two complete sets of basis vectors in the abstract space of all neutrino states. We will consider the role of neutrino mass in flavor-changing from two perspectives: (1) as an eigenvalue problem, and (2) from a rotation-of-axes perspective. Of course, the approaches are two ways of doing the same thing, but it may be instructive to examine both, not only for our understanding of neutrinos, but also to deepen our appreciation of what eigenvectors and eigenvalues are all about.

(1) Neutrino state eigenvectors

For simplicity, let's consider only two neutrino species. Begin with two mass eigenstates: neutrino state ν_1 carries a definite mass m_1, and neutrino state ν_2 carries a definite mass m_2. To reiterate, these neutrinos do not have a unique flavor; they are superpositions of two flavors.

The quantum wave function $\nu_n(t)$ of a freely propagating neutrino may be written with the usual quantum phase factor for a free particle of momentum *p* and energy *E* where, assuming for simplicity that the particle moves in only one spatial dimension *x*,

$$\nu_n(t) = A_n e^{i(px - Et)/\hbar} \tag{47}$$

where *n* = 1 or 2, and where A_n is a constant.

Since the neutrino masses are very small if not zero, we may assume these particles move through the lab frame with speeds $v \approx c$. Therefore $x \approx ct$. The momentum *p* can be extracted from the relativistic energy-momentum relation for a free particle,

$$E^2 - (pc)^2 = (mc^2)^2. \tag{48a}$$

Since *m* is small, using a binomial expansion gives

$$E \approx pc + \frac{(mc^2)^2}{2pc} \tag{48b}$$

and again, since *m* is small, we may use $p \approx E/c$ in the denominator of Eq. (48b):

$$E \approx pc + \frac{(mc^2)^2}{2E}. \tag{48c}$$

In solving Eq. (48c) for *p* and recalling $x \approx ct$, Eq. (47) becomes

$$\nu_n(t) \approx A_n e^{-i\omega_n t} \tag{49}$$

where

$$\omega_n \equiv \frac{(m_n c^2)^2}{2E\hbar}. \tag{50}$$

These two neutrino states, each of definite mass, can be arranged as a vector in the abstract space inhabited by two neutrino species. In the two-dimensional "mass basis" an arbitrary neutrino state is

$$|\nu_{mass}\rangle = \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} = \begin{pmatrix} A_1 e^{-i\omega_1 t} \\ A_2 e^{-i\omega_2 t} \end{pmatrix}. \tag{51}$$

Requiring all the mass to reside *somewhere* among neutrino species 1 and 2 gives the constraint

$$\langle \nu_{mass} | \nu_{mass} \rangle = 1 \tag{52a}$$

which with Eq. (51) yields

$$|A_1|^2 + |A_2|^2 = 1 \tag{52b}$$

so the A_n can be parameterized as

$$A_1 = \cos\theta \tag{52c}$$

$$A_2 = \sin\theta \tag{52d}$$

for some real number θ.

We are saying that the state of a freely propagating neutrino is an eigenstate of a Hamiltonian H_{free} whose eigenvalues include the masses m_n. That Hamiltonian would be

$$H_{free} = \hbar \begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{pmatrix} \tag{53}$$

because in the Schrödinger equation the states of Eq. (49) give $H_{free}|\nu_n\rangle = \hbar\omega_n|\nu_n\rangle$.

The neutrinos of definite flavor in the weak interactions, ν_e, ν_μ, and ν_τ, are not eigenstates of H_{free}. They can, however, be written as superpositions of the definite-mass neutrino eigenstates. Considering that we are using only two neutrino species, say ν_e and ν_μ [11], using Eqs. (51) and (52c-d) we may write for the electron neutrino

$$\nu_e(t) = a_1\nu_1(t) + a_2\nu_2(t)$$

$$= a_1 \cos\theta \, e^{-i\omega_1 t} + a_2 \sin\theta \, e^{-i\omega_2 t} \tag{54a}$$

and for the muon neutrino

$$\nu_\mu(t) = b_1 \cos\theta \, e^{-i\omega_1 t} + b_2 \sin\theta \, e^{-i\omega_2 t} \tag{54b}$$

where a_n and b_n are constants.

These neutrino states, along with $\nu_\tau(t)$, are flavor eigenstates of the weak-interaction Hamiltonian. The weak interactions can preserve flavor, such as

$$\mu^+ + \mu^- \to \nu_\mu + \bar{\nu}_\mu \tag{55a}$$

or they can change flavor without violating any lepton flavor conservation laws (the particle and antiparticles of the same flavor have opposite-sign lepton numbers), such as

$$\mu^+ + \mu^- \to \nu_e + \bar{\nu}_e. \tag{55b}$$

For generic interactions that allow both flavor-preserving and flavor-changing events, we can parameterize the neutrino interaction Hamiltonian (still in two-dimensional neutrino state space) as

$$H_{int} = \begin{pmatrix} H_{ee} & H_{e\mu} \\ H_{\mu e} & H_{\mu\mu} \end{pmatrix} \equiv \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}. \tag{56}$$

These matrix elements will depend on the weak force parameters such as the Fermi coupling constant G_F. The diagonal elements describe weak interactions that preserve lepton flavor, such as the reaction of (55a) and scattering events like $\nu_e + e^- \to \nu_e + e^-$. The off-diagonal elements describe weak interactions where the incoming and outgoing neutrinos have different flavors, such as reaction (55b) or $\nu_e + e^+ \to \nu_\mu + \mu^+$. But note that we have seen this sort of Hamiltonian before.

To find out what the a_n and b_n in Eqs. (54) are, the eigenvalue problem for H_{int} falls readily to hand, because its eigenvectors are the electron and muon neutrinos.

To proceed, for the Hamiltonian of Eq. (56) we parameterize its eigenstates as

$$|\nu_{int}\rangle = \begin{pmatrix} p \\ q \end{pmatrix} e^{-i\lambda t/\hbar} \tag{57}$$

where p and q are independent of time. Now the Schrödinger equation becomes

$$H_{int}|\nu_{int}\rangle = \lambda|\nu_{int}\rangle. \tag{58}$$

Our task is to find the eigenvalues λ through the application of the theorem of alternatives and then use these eigenvalues in Eq. (58) to find the eigenvectors normalized to unit magnitude. But this is mathematically identical to the ammonia molecule problem, so we have done this before. It follows that $\lambda = \alpha \pm \beta$, and the corresponding normalized time-independent portion of the eigenvectors of Eq. (51) are

$$\begin{pmatrix} p \\ q \end{pmatrix}_{\lambda=\alpha+\beta} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{59a}$$

and

$$\begin{pmatrix} p \\ q \end{pmatrix}_{\lambda=\alpha-\beta} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{59b}$$

Let us suppose that the $\alpha + \beta$ eigenvalue belongs to the electron neutrino. This means that Eqs. (59a) and (54a) at t = 0 are the same state, so that

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a_1 \cos\theta \\ a_2 \sin\theta \end{pmatrix} \tag{60}$$

and therefore $\frac{1}{\sqrt{2}} = a_1\cos\theta$ and $\frac{1}{\sqrt{2}} = a_2\sin\theta$, suggesting as *a* solution (not *the* unique solution, remember the theorem of alternatives) $a_2 = a_1 \equiv a$, which then requires $\sin\theta = \cos\theta = 1/\sqrt{2}$ and $\theta = \pi/4$. Restoring the time dependence by letting t be nonzero in Eq. (54a), we have the electron neutrino wave function expressed as superpositions of the mass eigenstate wave functions:

$$\nu_e(t) = \frac{a}{\sqrt{2}}(e^{-i\omega_1 t} + e^{-i\omega_2 t}). \tag{61a}$$

Carrying out the same procedure by identifying the $\alpha - \beta$ eigenvalue with the muon-type neutrino produces

$$\nu_\mu(t) = \frac{b}{\sqrt{2}}(e^{-i\omega_1 t} - e^{-i\omega_2 t}). \tag{61b}$$

Given the $1/\sqrt{2}$ in both flavor eigenstates, it appears that in the abstract space of neutrino states the flavor eigenstates are rotated by 45 degrees relative to the mass eigenstates. Let

$$\omega_2 = \omega_1 + \delta. \tag{62}$$

Then the flavor eigenstates can be written

$$\nu_e(t) = \sqrt{2}\, a\, e^{-i(\omega_1 + \delta/2)t} \cos\left(\frac{\delta t}{2}\right) \tag{63a}$$

$$\nu_\mu(t) = \sqrt{2}\, ib e^{-i(\omega_1 + \delta/2)t} \sin\left(\frac{\delta t}{2}\right). \tag{63b}$$

The probability of a neutrino being an electron neutrino at time t is

$$P_e(t) = |\nu_e|^2 = 2a^2\cos^2\left(\frac{\delta x}{2c}\right) \tag{64a}$$

and the probability of it being a muon neutrino at time t is

$$P_\mu(t) = |\nu_\mu|^2 = 2b^2\sin^2\left(\frac{\delta x}{2c}\right). \tag{64b}$$

If these were the only two neutrino flavors, then as long as the neutrino exists,

$$1 = P_e(t) + P_\mu(t) = 2(a^2 + b^2) \tag{65}$$

or $a^2 + b^2 = ½$, which allows these coefficients to be parameterized as $a = \frac{1}{\sqrt{2}}\cos\phi$ and $b = \frac{1}{\sqrt{2}}\sin\phi$ for some angle ϕ to be fit to data.

(2) Rotation of axes

Return to the concept that any two neutrino flavor states are a superposition of states of definite mass:

$$\begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}\begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \tag{66}$$

which can be abbreviated as

$$|\nu_{flavor}\rangle = \Lambda|\nu_{mass}\rangle \tag{67}$$

where

$$|\nu_{flavor}\rangle = \begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} \tag{68a}$$

$$|\nu_{mass}\rangle = \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \tag{68b}$$

and Λ is the square matrix in Eq. (66). So long as the neutrino exists—whatever its flavor and mass—we require

$$\langle\nu_{flavor}|\nu_{flavor}\rangle = \langle\nu_{mass}|\nu_{mass}\rangle. \tag{69}$$

The adjoint (denoted with †, the transpose and complex conjugate) of Eq. (66) gives

$$\langle\nu_{flavor}| = \langle\nu_{mass}|\Lambda^\dagger. \tag{70}$$

Now Eq. (69) becomes

$$\langle \nu_{mass} | \Lambda^{\dagger}\Lambda | \nu_{mass} \rangle = \langle \nu_{mass} | \nu_{mass} \rangle \qquad (71)$$

which requires

$$\Lambda^{\dagger}\Lambda = \tilde{1} \qquad (72a)$$

(one says that Λ is "unitary"), which in detail requires

$$\begin{pmatrix} a^*{}_1 & b^*{}_1 \\ a^*{}_2 & b^*{}_2 \end{pmatrix}\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \qquad (72b)$$

This condition implies

$$|a_1|^2 + |a_2|^2 = 1 \qquad (72c)$$

$$|b_1|^2 + |b_2|^2 = 1 \qquad (72d)$$

and

$$a^*{}_2 a_1 + b_1 b^*{}_2 = 0. \qquad (72e)$$

All of these constraints are consistent with

$$\Lambda = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \qquad (73)$$

Now Eq. (66), informed by Eq. (73) and with Eq. (49) showing the time dependence explicitly, becomes

$$\begin{pmatrix} \nu_e(t) \\ \nu_\mu(t) \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \nu_1(0)e^{-i\omega_1 t} \\ \nu_2(0)\, e^{-i\omega_2 t} \end{pmatrix}. \qquad (74)$$

The transformation from neutrino states of definite mass to states of definite flavor is merely a rotation of axes in the abstract two-dimensional space of neutrino states.

Our goal is to predict the probability that a neutrino which begins its life at time $t = 0$ as an electron neutrino will turn into a muon neutrino at time $t > 0$. In other words, we intend to calculate $|\langle \nu_\mu(t) | \nu_e(0) \rangle|^2$. To prepare the way, the time dependence of the mass eigenstates can be factored out of Eq. (74) as follows:

$$\begin{pmatrix} \nu_e(t) \\ \nu_\mu(t) \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} e^{-i\omega_1 t} & 0 \\ 0 & e^{-i\omega_2 t} \end{pmatrix}\begin{pmatrix} \nu_1(0) \\ \nu_2(0) \end{pmatrix}. \qquad (75a)$$

This gives, at $t = 0$, the relation

$$\begin{pmatrix} \nu_e(0) \\ \nu_\mu(0) \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \nu_1(0) \\ \nu_2(0) \end{pmatrix} \qquad (75b)$$

[which could be shown from Eq. (74), but the matrix with the exponential elements will prove useful]. In other words,

$$|\nu_{flavor}(0)\rangle = \Lambda |\nu_{mass}(0)\rangle \qquad (76a)$$

from which it follows that

$$\Lambda^{-1}\, |\nu_{flavor}(0)\rangle = |\nu_{mass}(0)\rangle. \qquad (76b)$$

From Eq. (72a) we see that $\Lambda^{-1} = \Lambda^{\dagger}$, therefore

$$\Lambda^{\dagger} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \qquad (77)$$

Now we can write $|\nu_{flavor}(t)\rangle$ in terms of $|\nu_{flavor}(0)\rangle$. Putting Eqs. (75a), (76b), and (77) together, we have

$$\begin{pmatrix} \nu_e(t) \\ \nu_\mu(t) \end{pmatrix} = \Lambda \begin{pmatrix} e^{-i\omega_1 t} & 0 \\ 0 & e^{-i\omega_2 t} \end{pmatrix} \Lambda^{\dagger} \begin{pmatrix} \nu_e(0) \\ \nu_\mu(0) \end{pmatrix}. \qquad (78)$$

Writing out the matrix multiplication is a bit of work; to simplify the result it helps to use Eq. (62), $\omega_2 = \omega_1 + \delta$. The top component of Eq. (78) is, in Dirac bracket notation,

$$|\nu_e(t)\rangle = e^{-\omega_1 t}\left[U|\nu_e(0)\rangle - i\sin(2\theta)\sin\left(\frac{\delta t}{2}\right)|\nu_\mu(0)\rangle\right] \qquad (79a)$$

where $U \equiv \cos^2\theta + e^{i\delta t}\sin^2\theta$. The bottom component of Eq. (78) is

$$|\nu_\mu(t)\rangle = e^{-\omega_2 t}\left[-i\sin 2\theta \sin\left(\frac{\delta t}{2}\right)|\nu_e(0)\rangle + U|\nu_\mu(0)\rangle\right]. \qquad (79b)$$

Let a neutrino start out as an electron neutrino at $t = 0$. The amplitude for it to become a muon neutrino at time $t > 0$ follows by multiplying $\langle \nu_\mu(t)|$ of Eq. (79b) with $|\nu_e(0)\rangle$ and using the orthonormality of the flavor states; therefore,

$$\langle \nu_e(0) | \nu_\mu(t) \rangle = i e^{i\zeta t}\sin(2\theta)\sin\left(\frac{\delta t}{2}\right) \qquad (80a)$$

where $\zeta = \omega_1 + \frac{\delta}{2}$. The corresponding probability follows,

$$P_{e\to\mu}(t) = \sin^2(2\theta)\sin^2\left(\frac{\delta t}{2}\right). \qquad (80b)$$

Similarly, the probability for a neutrino that is a muon neutrino at $t = 0$ to become an electron neutrino at $t > 0$ is

$$P_{\mu\to e}(t) = |\langle \nu_e(t)|\nu_\mu(0)\rangle|^2$$
$$= \sin^2(2\theta)\sin^2\left(\frac{\delta t}{2}\right). \tag{80c}$$

If $\delta \neq 0$, in other words, if $m_1 \neq m_2$, then *electron neutrinos and muon neutrinos can change back and forth into each other—neutrino oscillations occur* [11]. The spatial period of the oscillation—how far a neutrino travels before it changes from one flavor to the other—can be found by using $t = x/c$ to convert the neutrino's travel time to the distance traveled. The wavelength λ of the oscillation can then be read off the phase in Eqs. (80), recognizing $\delta/2c$ as a wavenumber $k = 2\pi/\lambda$, which gives

$$\frac{\delta}{2c} = \frac{2\pi}{\lambda}. \tag{81a}$$

From Eq. (62), $\delta = \omega_2 - \omega_1$, which by Eq. (50) says

$$\delta = \frac{1}{2E\hbar}[(m_2c^2)^2 - (m_1c^2)^2]. \tag{81b}$$

Now Eq. (81a) gives, for the distance between neutrino flavor-changing events,

$$\lambda = 8\pi\hbar c\left(\frac{E}{(m_2c^2)^2 - (m_1c^2)^2}\right)$$
$$\approx 1 \times 10^{-5}\ \mathrm{m\ eV}\left(\frac{E}{(m_2c^2)^2 - (m_1c^2)^2}\right). \tag{81c}$$

Data for electron- and muon-type neutrinos says $(m_2c^2)^2 - (m_1c^2)^2 \approx 7.5 \times 10^{-5}\ \mathrm{eV}^2$. At $E = 20\ \mathrm{MeV}$, these numbers give in Eq. (81c) a distance $\lambda_{e\leftrightarrow\mu}$ = 1300 km (780 miles) for electron-muon oscillations. The value of $(m_2c^2)^2 - (m_1c^2)^2$ for muon- and tau-type neutrinos is $2.4 \times 10^{-3}\ \mathrm{eV}^2$, which at 20 MeV gives $\lambda_{\mu\leftrightarrow\tau}$ = 32 m. Data suggest that the mixing angle θ for electron-muon neutrino oscillations is about 33.9° and about 45° for muon-tau oscillations [12].

How to Catch a Neutrino

The Reines-Cowan and Davies experiments were sensitive to electron-type neutrinos. Other options were soon forthcoming. In 1974 Howard Georgi and Sheldon Glashow and others predicted proton decay as a consequence of "grand unified" theories, which put quarks and leptons in a common family and allowed transitions between them. These theories predict a proton lifetime on the order of 10^{31} years. We can't wait for 10^{31} years to see if a proton decays in the reaction $p \to e^+ + \pi^0$, but in a collection of 10^{31} protons, there should be about one decay per year. A thousand tons of matter would have about 5×10^{32} protons and, if the theory is correct, about 50 of them per year should decay [13]. Assemble a thousand tons of transparent matter (pure water), surround it with photomultiplier tubes, and watch for the Cherenkov radiation (light emitted by a charged particle moving at the speed *v,* where *c/n* < *v* < *c* with *n* the medium's refractive index), a "shock wave" cone of light that can be detected by photomultiplier tubes.

Several such detectors were built in the early 1980s, but the idea was not new. In 1954 Frederick Reines, Clyde Cowan, and Maurice Goldhaber, using part of the detector with which they detected the antineutrino two years later, put a lower bound of 10^{22} years on proton decay, and a 1974 proposal by Reines and William Kropp presciently envisioned a 10-kiloton proton decay detector—which was turned down for "lack of theoretical motivation" at that time [14]. Subtracting neutrino backgrounds offered a serious challenge to these experiments; for example, elastic scattering of neutrinos by electrons, $\nu_e + e^- \to \nu_e + e^-$, could produce an electron with enough energy to emit Cherenkov radiation.

So far proton decay has not been observed. But with such massive detectors in place that are sensitive to neutrino events, serendipitous results ensued. In 1998 a group working with a detector deep in a mine in Kamiokande, Japan, first published evidence for oscillations in "atmospheric neutrinos" [15], neutrinos produced when cosmic rays from the sun collide with nuclei in the upper atmosphere to produce muons and muon neutrinos. The discovery was made by "Super-Kamiokande," which uses Earth's diameter as a baseline: solar or atmospheric neutrinos pass through us from above in the daytime and come up through the floor at night. Super-Kamiokande reported, "The number of the upward going neutrinos was only half of the number of the down going neutrinos. This is because the muon neutrinos passing through the earth turn into tau neutrinos" [14] (recall that the muon-tau oscillation length is quite short). Super-Kamiokande produced the first hard evidence for $\mu - \tau$ neutrino oscillations in *atmospheric* neutrinos but not conclusive verification of oscillations in *solar* neutrinos.

In 1984 the neutrino detection story took another turn when Herb Chen pointed out the advantages of using hydrogen-2 as the detector medium [16]. In a charged current interaction (similar to the Homestake reaction, but with deuterons instead of chlorine nuclei), the neutrino collides with the neutron in hydrogen-2, turning it into a proton and electron: $\nu_e + \mathrm{H}_1^2 \to p + p + e^-$. Since *E* < 20

MeV for solar neutrinos and the muon mass = 106 MeV/c^2, only electron neutrinos participate here, although this reaction is detectable since the electron carries 5–15 MeV of energy. But there is also a neutral current interaction that can proceed with any lepton flavor:

$$\nu_\rho + \mathrm{H}_1^2 \rightarrow n + p + \nu_\rho \qquad (82)$$

where $\rho = e, \mu$, or τ. This possibility led to the founding of the Solar Neutrino Observatory (SNO) near Sudbury, Ontario. SNO is a 12-meter-diameter acrylic sphere containing 1000 tons of heavy water, surrounded by photomultiplier tubes to detect Cherenkov radiation. The heavy water was loaned to the experiment by Atomic Energy of Canada. The sphere containing heavy water was surrounded by another sphere containing ordinary water.

In a reaction (82) the neutron that emerges can react with a second deuteron to make helium-3 and a 6 MeV photon: $n + \mathrm{H}_1^2 \rightarrow \mathrm{He}_3^3 + \gamma\ (6\ \mathrm{MeV})$. One could collect the helium (analogous to how the Homestake experiment collected the noble gas argon), but the photons are detected immediately by the array of photomultiplier tubes. Neutrons that escape the heavy water sphere enter the shell of ordinary water, where some of them collide with hydrogen-1 to drive the reaction $n + \mathrm{H}_1^1 \rightarrow \mathrm{H}_1^2 + \gamma(2.2\ \mathrm{MeV})$ to produce more signals for the photomultiplier tubes. The SNO results of 2001 were firm evidence for neutrino oscillations in solar neutrinos [17].

The 2015 Nobel Prize in Physics was awarded to Arthur McDonald, director of SNO, and Takaaki Kajita, leader of the group at Kamiokande, for the observation of neutrino oscillations. The humble coupled oscillator touches a lot of physics!

References

[6] The energies released in beta decays tend to be a few MeV. Compared to the electron's mass of 0.511 MeV/c^2, one might want to use the relativistic expression for the electron's kinetic energy when relating it to momentum, whereas the neutron and proton masses are ≈ 1 GeV/c^2, so Newtonian kinetic energy can be used for them.

[7] "The Reines-Cowan Experiments: Detecting the Poltergeist," *Los Alamos Sci*. 25 (1997): 4–27. See https://library.lanl.gov/cgi-bin/getfile?00326606.pdf. The title refers to experiments in the plural because it also gives a thorough description of Reines's original idea (1951) to bury a neutrino detector deep under ground zero of a nuclear bomb test site.

[8] The three families or "flavors" of quarks are the "up" *u* and "down" *d;* "charm" *c* and "strange" *s*; and "top" *t* and "bottom" *b*. Ordinary matter (what we see on the Periodic Table) is made of "first-generation" quarks and leptons: *u* and *d* quarks (proton = *uud*, neutron = *ddu*), and the leptons e^- and ν_e. The second generation consists of *c, s*, and the muon and ν_μ; the third generation of *t, b,* and the tau and ν_τ.

[9] André de Gouvêa and Petr Vogel, "Lepton Flavor and Number Conservation, and Physics Beyond the Standard Model," arXiv:1303.4097v2[hep-ph], 29 Mar 2013, 1–27. A lot depends on whether neutrinoless double beta decay occurs.

[10] Bruno Pontecorvo, "Mesonium and Anti-Mesonium," *Zh. Eksp. Teor. Fiz*. 33 (2 Feb. 1957): 549–551. Translated and reproduced in *Sov. Phys. JETP* 6 (2 Feb. 1957): 429–431.

[11] The most common neutrino oscillations seem to occur between muon and tau neutrinos.

[12] Data from K. A. Olive et al., *Particle Data Group* (2010). These figures (rounded) are also presented in Paul Tipler and Robert Llewellyn, *Modern Physics* (San Francisco: Freeman, 2008), 610.

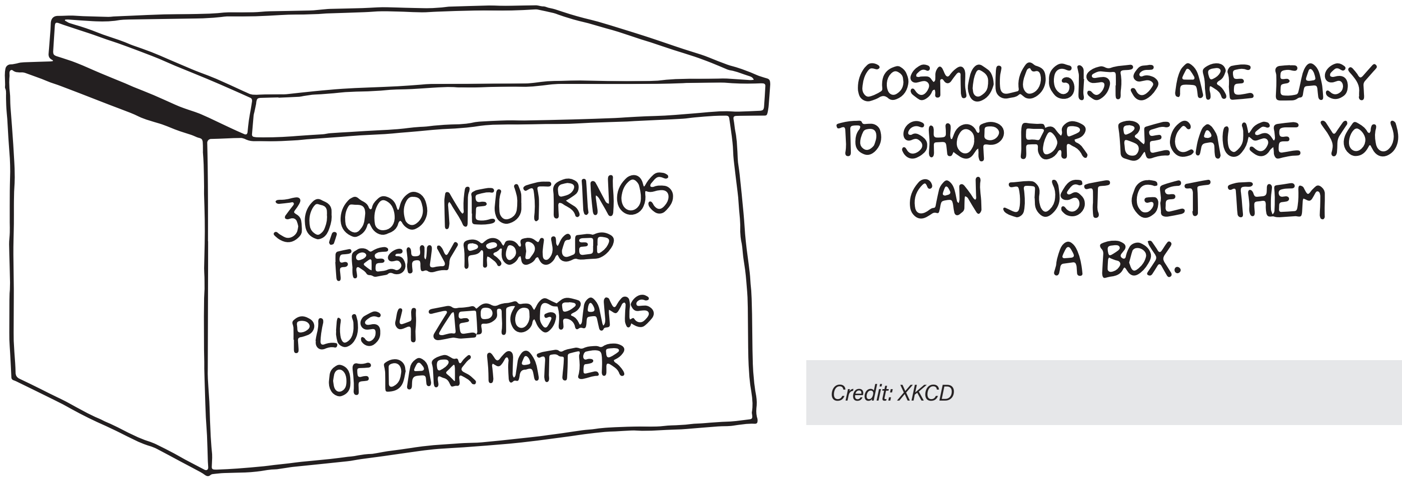
[13] Howard Georgi and Sheldon Glashow, "Unity of All Elementary Particle Forces," *Phys. Rev. Lett.* 32 (1974): 438; Howard Georgi, "A Unified Theory of Elementary Particles and Forces," *Sci. Am.* (April 1981): 48–63.

[14] "Underground Experiments Will Look for Proton Decay," *Physics Today* (Jan. 1980): 17–19.

[15] Y. Fukuda et al., Super-Kamiokande Collaboration, "Evidence for Oscillation of Atmospheric Neutrinos," *Phys. Rev. Lett.* 81, no. 8 (1998): 1562–1567; Super-K website: http://www-sk.icrr.u-tokyo.ac.jp/sk/sk/neutrino-e.html.

[16] Herbert Chen, "Direct Approach to Resolve the Solar-Neutrino Problem," *Phys. Rev. Lett.* 55, no. 14 (1985): 1534–1536.

[17] SNO's homepage: https://falcon.phy.queensu.ca/SNO/. See also https://en.wikipedia.org/wiki/Sudbury_Neutrino_Observatory.



Credit: XKCD