A Motorcycle or Bicycle as a Gyroscope (Sort Of)

Part II: Bike stability and the effects of gyroscopic action

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In the Spring installment of this article, we considered how the wheels of a bicycle or motorcycle (bike) make the machines a kind of gyroscope. We also examined how the two spinning wheels are not as essential to the bike's stability as we might think, and we examined the importance of the castor, the distance between the steering axis–ground intersection and the front tire's ground contact point, in stability. Recall that the castor is denoted δ (Fig. 2).

δ

δ

Figure 2 (from part I): Showing the castor distance on a 1994 Kawasaki Vulcan (top) and a 1962 J.C. Higgins Flightliner (bottom). Their front wheels have been deliberately set straight ahead. All photos and images by Dwight E. Neuenschwander unless otherwise noted.

Now let us review the essential dimensions that concern us. Recall that Fig. 4 shows a schematic: a denotes the bike's wheelbase; Z is the point on the ground directly below the CM with the bike upright; b denotes the distance between Z and the rear tire's contact point with the ground; h is the height of the CM above the ground; and the castor distance δ is shown with an idealized vertical steering axis.

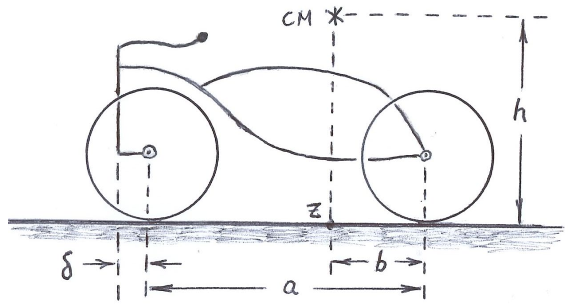

Figure 4 (from part I): View of bike from the side, showing the distances a, b, h, and δ.

We then considered the bike moving through a turn, its path the arc of a circle of radius R. From Fig. 5a, an overhead view of the bike, we let α be the angle relative to the bike frame through which the front wheel is turned; let η be the angle relative to an arbitrary fixed direction through which the frame has turned as the bike moves along the arc; and let $\hat{n}$ denote a unit vector normal to the bike's frame and pointing towards the center of curvature of the circular arc. From Fig. 5b, a view of the bike from behind it, we let θ be the lean angle of the bike from the vertical. For dynamic variables we let m denote the mass of the bike and rider, and g the magnitude of the gravitational field.

Now let's move on to consider bike stability and the effects of gyroscopic action.

On Stability

As discussed in Part I, Newton's second law in rotational form says that net torque about the steering axis on the handlebars-fork-front wheel system—torques due to lean and friction—produces a change in the vertical component of angular momentum according to

$$\frac{mb\delta}{a}\left(g\theta - \frac{v^2\alpha}{a}\right) = \frac{d}{dt}\left(I_f\dot{\alpha} - I_o\omega \sin \sin \theta\right) \quad (22)$$

where I_f denotes the moment of inertia of the handlebar-fork-front wheel assembly and I_o denotes the front wheel's moment of inertia about its axle with ω the wheel's angular velocity.

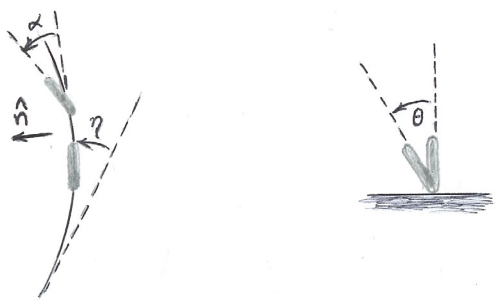

Figure 5 (from part I): View of bike from above, showing angles α and η and the unit vector $\hat{\boldsymbol{n}}$ (left). View of bike from behind it, showing the angle θ (right).

Let us return to Eq. (22) and include the front wheel's moment of inertia about its axle. With $\sin\theta \approx \theta$ and noting that $\omega = v/r$ where *r* denotes the front wheel's outer radius, Eq. (22) becomes

$$\ddot{\alpha} - \mu\left(\theta - \frac{v^2}{ga}\alpha\right) - \gamma v\dot{\theta} = 0 \quad (25)$$

where we have defined a gyroscopic factor γ

$$\gamma \equiv I_o/I_f\, r \qquad (26)$$

and a castor factor

$$\mu \equiv mgb\delta/aI_f\,. \qquad (27)$$

Suppose the bike travels straight and upright initially, where $\alpha = 0$ and $\theta = 0$. If at some moment $\dot{\theta}$ becomes nonzero (while α and θ are still approximately zero) then by Eq. (25) it follows that $\ddot{\alpha} \neq 0$ and the bike turns in the direction of the lean *because* of the gyroscopic effect, $\gamma \neq 0$. Let us also recall Eq. (10) from Part I, which gives the component of the gravitational force in the $\hat{\boldsymbol{n}}$ direction. Assuming θ to be small and neglecting friction, the $\hat{\boldsymbol{n}}$ component of Newton's Second Law says, with some rearrangement from (10):

$$\ddot{\theta} - \frac{g}{h}\left(\theta - \frac{v^2}{ga}\alpha\right) + \frac{bv}{ha}\dot{\alpha} = 0. \quad (28)$$

Eqs. (25) and (28), which reveal dynamic coupling between α and θ, will be our working equations in the considerations that follow.

Since Jones's experiments with modified bicycles show that gyroscopic effects are not dominant factors for bike stability, let us examine bike stability in the extreme case of $\gamma = 0$, which will be followed by the $\gamma \neq 0$ case in order to see how large a role the gyroscopic factor *does* play.

Case 1: $\gamma = 0$
With zero gyroscopic effects Eq. (25) becomes

$$\ddot{\alpha} = \mu\left(\theta - \frac{v^2}{ga}\alpha\right). \qquad (29)$$

Being proportional to the weight of bike and rider, μ is typically quite large. Consequently, if $\theta - \frac{v^2\alpha}{ga} > 0$ (e.g., the rider leans with the front wheel initially pointed straight ahead) then α grows until $\theta - \frac{v^2\alpha}{ga} = 0$. Because $\theta - \frac{v^2\alpha}{ga}$ will eventually become zero, let us assume

$$\theta = \frac{v^2}{ga}\alpha. \qquad (30)$$

With this constraint used in Eq. (28) the equation of motion for θ reduces to

$$\ddot{\theta} + \kappa\dot{\theta} = 0 \qquad (31)$$

where

$$\kappa \equiv bg/hv. \qquad (32)$$

Denoting the initial lean angular velocity as $\dot{\theta}(0) = W_o$, Eq. (31) integrates to

$$\theta(t) = \frac{W_o}{\kappa}(1 - e^{-\kappa t}) \qquad (33)$$

from which we obtain the asymptotic lean angle

$$\theta(\infty) = W_o/\kappa. \qquad (34)$$

Therefore, by Eqs. (30) and (32),

$$\alpha(\infty) = W_o ha/bv. \qquad (35)$$

Note that $\theta(\infty)$ and $\alpha(\infty)$ have the same sign—the front wheel turns in the direction of the lean. The leaning bike stabilizes itself by moving in a circular path of some radius *R*. To determine *R*, when the bike has traveled a distance along the circular arc equal to its wheelbase *a*, the bike frame has rotated through the angle $\alpha = \frac{a}{R}$ (see Eq. 6 in part 1), which by Eq. (35) gives for the circle's radius

$$R = \frac{bv}{hW_o}. \qquad (36)$$

All of this occurs by assuming $\gamma = 0$, i.e., with *no gyroscopic help whatsoever* from the wheels! Clearly the gyroscopic action is not *essential* for turning a motorcycle or a bicycle.

While still in the $\gamma = 0$ case, let us lift the restriction of Eq. (30) and allow $\theta - \frac{v^2}{ga}\alpha \neq 0$. Eq. (25) now becomes

$$\ddot{\alpha} + \frac{\mu v^2}{ga}\alpha = \mu\theta. \qquad (37)$$

This resembles the equation of a driven undamped harmonic oscillator. According to Eq. (37), if the handlebars are turned at a moment when θ passes through zero, then at that moment the front wheel may begin oscillating, because Eq. (37) then describes a simple harmonic oscillator. In that event we have

$$\alpha(t) \approx \alpha_o \cos(\omega t) \qquad (38)$$

with angular frequency ω,

$$\omega = \sqrt{\frac{\mu v^2}{ga}}. \qquad (39)$$

Lowell and McKell tabulate the necessary parameters for a particular bicycle (model not given) and rider.[2] For instance, their bike had $a = 1.0$ m, $b = 0.33$ m, $h = 1.5$ m, $r = 0.33$ m, $m = 80$ kg. They cite $\mu = 133\ s^{-2}$, and mention in their acknowledgments that this "Jones couple" was "rather tricky" to measure, I presume because of I_f. Their bicycle data predicts an oscillation frequency $\frac{\omega}{2\pi} \approx 2$ Hz at $v = 3.5$ m/s—low enough for an alert rider to easily make necessary corrections. But the existence of these oscillations opens the door to the possibility of a runaway oscillation, the notorious speed wobble that cyclists (especially racers) encounter in some circumstances.

How does the front wheel's oscillation affect the lean angle θ? Eq. (28), repeated here with some rearrangement, says

$$\ddot{\theta} - \frac{g}{h}\theta = -\frac{v}{ha}(v\alpha + b\dot{\alpha}) \qquad (40)$$

where *h* is the CM height, as described in part 1 of the article. When Eq. (38) holds and α is oscillatory, then we expect θ to also be oscillatory but possibly phase-shifted from α, because if Eq. (38) holds then Eq. (40) becomes, within a phase factor,

$$\ddot{\theta} - \frac{g}{h}\theta \approx -\frac{v\alpha_o}{ha}[v\cos(\omega t) - b\omega\sin(\omega t)]. \qquad (41)$$

The complementary solution of Eq. (41) suggests the possibility of a runaway oscillation $\theta \sim e^{+t\sqrt{\frac{g}{h}}}$ which, fortunately in most riding, does not often arise. Lowell and McKell remark,

"To summarize, if gyroscopic effects are ignored, the bicycle is almost self-stable. A perturbation tending to push it over results, in the first approximation, merely in the bicycle entering a curved path. However, the bicycle is <u>*unstable*</u> *in the sense that oscillations in* α *tend to grow. In practice, the oscillatory instability would probably not matter; growth is very slow and it is possible that the oscillations would not be noticeable if the rider were anything other than completely inert."* [2]

Though not *essential* to stability, do gyroscopic effects *contribute* to stability, perhaps to damp out front wheel oscillations? Gyroscopic effects have, surely, *some* effect. Let us probe further by allowing γ to be nonzero.

Case 2: $\gamma \neq 0$

Let us return to Eq. (25) with $\gamma \neq 0$. Because the rider does not normally *suddenly* jerk the handlebars to a large angle (doing so would be disastrous), $\ddot{\alpha}$ is very small; let us approximate it as $\ddot{\alpha} = 0$, and allow that $\theta - \frac{v^2}{ga}\alpha \neq 0$. Now Eq. (25) may be written

$$\alpha = \frac{ga}{v^2}\left(\theta + \frac{\gamma v}{\mu}\dot{\theta}\right). \qquad (42)$$

Differentiating with respect to time yields:

$$\dot{\alpha} = \frac{ga}{v^2}\left(\dot{\theta} + \frac{\gamma v}{\mu}\ddot{\theta}\right). \qquad (43)$$

Insert Eqs. (42) and (43) for α and $\dot{\alpha}$ into Eq. (28), which becomes

$$\ddot{\theta}\left(1+\frac{\gamma b g}{h\mu}\right)+\frac{gb}{hv}\left(1+\frac{\gamma v^2}{\mu b}\right)\dot{\theta}=0. \quad (44)$$

Eq. (44) integrates to

$$\theta(t)=\frac{W_o}{\kappa'}\left(1-e^{-\kappa' t}\right) \qquad (45)$$

where, in terms of the κ of Eq. (32),

$$\kappa'=\kappa\left(\frac{1+\frac{\gamma v^2}{\mu b}}{1+\frac{\gamma b g}{h\mu}}\right)\equiv \kappa\zeta. \quad (46)$$

Notice that $\kappa'=\kappa$ when $\gamma=0$. If $\zeta>1$ then θ damps to its asymptotic value faster than it does with $\gamma=0$, which means gyroscopic effects *enhance* stability. In Eq. (46) ζ exceeds 1 if $\frac{\gamma v^2}{\mu b}>\frac{\gamma b g}{h\mu}$, or $v>b\sqrt{g/h}$. This inequality is satisfied for v > 1 m/s for the bicycle parameters cited by Lowell & McKell,[2] a minimum velocity easily attained. But if $\zeta<1$, i.e., if $v<b\sqrt{g/h}$, then the oscillations damp more slowly than they would if $\gamma=0$.

Turning to the effect of γ on oscillations when $\ddot{\alpha}$ is not negligible, let us recall our working equations for $\ddot{\alpha}$ and $\ddot{\theta}$, Eqs. (25) and (28), repeated here for convenience:

$$\ddot{\alpha}+\omega^2\alpha=\mu\theta+\gamma v\dot{\theta} \qquad (47)$$

where ω^2 is given by Eq. (39), and

$$\ddot{\theta}-\frac{g}{h}\theta=-\frac{v}{ha}(v\alpha+b\dot{\alpha}). \quad (48)$$

If it were not for some contrasting minus signs, it would appear that α and θ are proportional and in phase for all situations in which Eqs. (47) and (48) apply. But despite differing signs, in the complementary solutions of these differential equations (when the right-hand sides are zero) we see intimations of a runaway oscillation: α could be oscillatory, $\alpha_c\sim\cos(\omega t)$, and θ could be exponential, $\theta_c\sim e^{\pm t\sqrt{\frac{g}{h}}}$.

Lowell and McKale carried out numerical solutions of Eqs. (47) and (48). They presented their results in graphs of θ vs. *t*; qualitative schematic sketches of them are shown in Fig. 10. The dotted curves include no gyroscopic effects ($\gamma=0$) and the solid curves consider nonzero γ; the red curves are for a faster speed than the black curves. These authors note that "Gyroscopic action is stabilizing in the sense that it results in a smaller (mean) value of θ, but destabilizing in the sense that it enhances the oscillatory instability."[2]

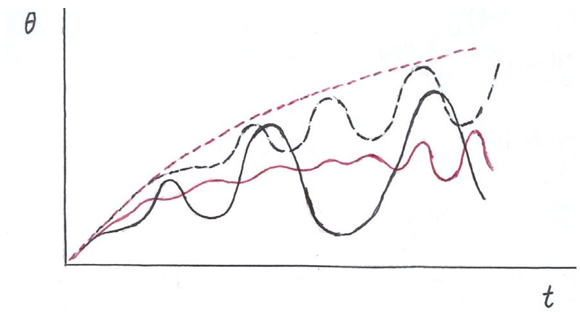

Figure 10: Schematic of data from Fig. 3 of Ref. 2 showing $\theta(t)$ for red (fast) and black (slow) velocities. The dotted curves have $\gamma=0$ and the solid curves have $\gamma\neq 0$.

Let us see if we can extract an analytic approximation to this behavior. Write Eq. (47) as

$$\ddot{\alpha}-\gamma v\dot{\theta}+\omega^2\alpha=\mu\theta. \qquad (49)$$

Figure 1 (from part I): Jared Mees and his Indian Scout FTR 750 motorcycle in a controlled power slide during a flat-track race. Photo courtesy of David Hoenig, Flat Trak Fotos.

Consider a bike coming out of a left turn. During a normal left turn the bike leans to the left and the front wheel turns to the left, so that $\theta>0$ and $\alpha>0$. In coming out of the normal turn, θ and α both decrease to zero, so $\dot{\theta}<0$ and $\dot{\alpha}<0$. When the turn is completed, the bike goes straight and both angles have returned smoothly to zero (if there are no overshoots and oscillations). But in a controlled power slide like the one shown in Fig. 1, during the slide $\theta>0$ (leaning to the left) but $\alpha<0$ (front wheel turned to the right). In coming out of the slide and heading down the straightaway, the rider returns both angles back to zero. Thus $\dot{\theta}<0$ but $\dot{\alpha}>0$. To consider both cases of straightening the bike up when coming out of a turn—a normal turn or a power slide—it may be reasonable to assume that, within a phase shift,

$$\dot{\theta} = \pm k\dot{\alpha} \qquad (50)$$

for some $k > 0$. As the bike approaches $\theta \approx 0$ Eq. (49) becomes

$$\ddot{\alpha} \mp 2\beta\dot{\alpha} + \omega^2\alpha = 0 \qquad (51)$$

where

$$\beta \equiv k\gamma v/2. \qquad (52)$$

Parameterizing the solution as $\alpha(t) \sim e^{\sigma t}$ for some constant σ turns Eq. (51) into

$$\sigma^2 \mp 2\beta\sigma + \omega^2 = 0 \qquad (53)$$

and solving for σ gives

$$\alpha(t) \sim e^{\pm\beta t}\exp(\mp t\sqrt{\beta^2 - \omega^2}). \quad (54)$$

If $\beta^2 < \omega^2$, i.e. if $k\gamma v < \omega$, then $\sqrt{\beta^2 - \omega^2}$ becomes imaginary which makes $\exp(\mp t\sqrt{\beta^2 - \omega^2})$ sinusoidal, and in that case

$$\alpha(t) \sim e^{\pm\beta t}\cos(t\sqrt{\omega^2 - \beta^2}) \qquad (55)$$

which includes both possibilities of damped or runaway oscillations. By Eq. (50) a similar result holds for θ.

Having experienced a speed wobble or two myself (though not as spectacular as Don Castro's) here is my hypothesis on what may be happening in a speed wobble. As the rider comes out of the slide with θ and α approaching zero, if both angles overshoot their zeroes the rider may for a moment be leaning to the right with the front wheel pointing to the left. Of course, in any riding situation the rider makes corrections continually, but if the overshoots of θ and α are not sufficiently small, some combination of parameters can lead very quickly to an exponentially growing speed wobble (the + sign in the exponential of Eq. 54).

Tire shape also has an important effect on bike steering and stability. The cross-section of a car tire is approximately horizontal where it touches the road. But because a bike leans over in turns, the cross-sections of motorcycle and bicycle tires are rounded. Thus during the lean significant tire surface still contacts the road, contact so necessary for maintaining the frictional force between tire and road. In addition, the tires of a car or non-leaning bike roll like cylinders, but when a bike leans its tires behave more like rolling cones (Fig. 11)[1]—the trajectory turns in the direction of the cone's smaller end.

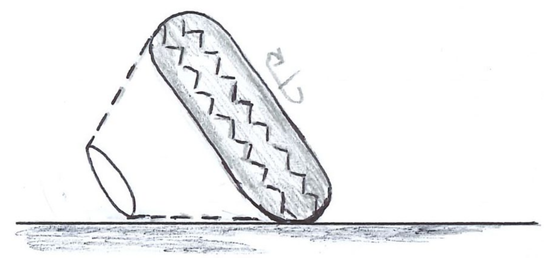

Figure. 11: Because of its rounded cross-section, a rolling leaning tire behaves like a cone, helping the bike turn.

When riding a bicycle or motorcycle, one must pay sharp attention to other traffic, road conditions, and being visible (assume you are invisible). However, when traffic is minimal and there are no woods next to the road that might conceal foraging deer, you can meditate on some interesting physics of the stability and steering of bikes. Ride safe! •

Figure 12: When leaning on the kickstand a bike's front wheel turns into the lean, as illustrated with a 2006 Yamaha Royal Star Midnight Venture near Pie Town, New Mexico.

Acknowledgment
I thank David Hoenig and Kathy Hoenig of Flat Trak Fotos for graciously granting permission to use the photo in Fig. 1.

This article offers a drastic revision of the Summer 2003 *Radiations* Elegant Connections article, which was ridiculously over-simplified.

Part I of this revised article is available on the *Radiations* website, *www.sigmapisigma.org/sigmapisigma/radiations/spring/2021/motorcycle-or-bicycle-gyroscope-sort.*

[1] A.J. Cox, "Angular momentum and motorcycle counter-steering: A discussion and demonstration," *Am. J. Phys*. **66** (11), 1018-1020.
[2] J. Lowell and H.D. McKell, "The stability of bicycles," *Am. J. Phys.* **50** (1982), 1106-1112.
[3] J. Fajans, "Steering in bicycles and motorcycles," *Am. J. Phys.* **68** (7), July 2000, 654-659.
[4] J. Higbie, "The Motorcycle as a Gyroscope," *Am. J. Phys.* **42**, 701-702 (1974).
[5] Thomas B. Greenslade Jr., "More bicycle physics," *The Physics Teacher*, Sept. 1983, 360-363.
[6] David E.H. Jones, "The stability of the bicycle," *Physics Today,* Sept. 2006, 51-56.
[7] In countersteering for a left turn, one can push the left handlebar forward, or pull the right handlebar backward, or both. Either way produces a clockwise initial torque, which is overcome by the counterclockwise torque due to sideways friction on the tire. For a right turn, push the right handlebar forward and/or pull the left handlebar backwards. These are very subtle gentle pushes, not yanks, but the response is immediate.